Tutorial 5

Transforming maximin problem to dual problem

We solve the above maximin problem in the following two steps.

1. Transform the maximin problem to a dual problem.

2. Use simplex method to solve the dual problem.

Step 1. Add a constant k to each entry of A so that every entry of A is positive.

Step 2. Let $x_i = \dfrac{p_i}{v}$, for $i = 1, 2, \cdots, m$.

Step 3. Solve the dual problem

$$\begin{aligned} \min \quad & g(\mathbf{x}) = \mathbf{x}1^T \\ \text{subject to} \quad & \mathbf{x}A \geq 1 \end{aligned}$$

Step 4. Suppose $\mathbf{x} = (x_1, x_2, ..., x_m)$ is an optimal vector of the dual problem and

$$d = g(\mathbf{x}) = x_1 + x_2 + ... + x_m$$

is the minimum value. Then,

$$\mathbf{p} = \frac{\mathbf{x}}{d}$$

is a maximin strategy for the row player and the value of the game matrix A is

$$v = \frac{1}{d} - k.$$

Simplex method

Simplex method is a method to solve the linear programming problems.

Given an $m \times n$ matrix A, two vectors $\boldsymbol{b} \in \mathcal{P}^m$, $\boldsymbol{c} \in \mathcal{P}^n$ and a number d, we consider *primal problem*

$$\begin{aligned} \max \quad & f(\boldsymbol{y}) = \boldsymbol{c}\boldsymbol{y}^T + d \\ \text{subject to} \quad & A\boldsymbol{y}^T \leq \boldsymbol{b}^T \end{aligned}$$

and the *dual problem*

$$\begin{aligned} \min \quad & g(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{b}^T + d \\ \text{subject to} \quad & \boldsymbol{x}A \geq \boldsymbol{c}. \end{aligned}$$

The key step of the simplex method is called the **pivoting operation**. Assume the tableau of the linear programming problem is given by

	y_1	$\cdots$	y_n	-1
x_1	a_{11}	$\cdots$	a_{1n}	b_1
$\vdots$	$\vdots$		$\vdots$	$\vdots$
x_m	a_{m1}	$\cdots$	a_{mn}	b_m
-1	c_1	$\cdots$	c_n	$-d$

Step 1. Find a position to start the pivoting operation.

If $c_j \leq 0$ for all j, then go to step 3. Otherwise, choose $j \in \{1, 2, \cdots, n\}$ such that $c_j > 0$.

If $a_{ij} \leq 0$ for all $1 \leq i \leq m$, the primal problem has no solution. Otherwise, pick $k \in \{1, 2, \cdots, m\}$ such that

$$\frac{b_k}{a_{kj}} = \min\{\frac{b_i}{a_{ij}} : a_{ij} > 0, i = 1, \cdots, m\}.$$

Step 2. Make pivoting operation as follows.

$$
\begin{array}{c|cc|}
 & y_k & y_l \\
\hline
x_i & a^* & b \\
x_j & c & d \\
\hline
 & &
\end{array}
\longrightarrow
\begin{array}{c|cc|}
 & x_i & y_l \\
\hline
y_k & \frac{1}{a} & \frac{b}{a} \\
x_j & -\frac{c}{a} & \frac{ad-bc}{a} \\
\hline
 & &
\end{array}
$$

Step 3. Continue Step 1 and Step 2 until $c_j \leq 0$ for all j. If the final result after pivoting operations is

$$
\begin{array}{c|cc|c}
 & x_i & y_l & -1 \\
\hline
y_j & & & e \\
x_k & & & f \\
\hline
-1 & g & h & -v
\end{array},
$$

then we can conclude that the optimal value of the primal problem is v and

$$
\begin{aligned}
x_i &= -g \quad y_{i+n} = 0 \\
y_l &= 0 \quad x_{l+m} = -h \\
y_j &= e \quad x_{j+n} = 0 \\
x_k &= 0 \quad y_{k+m} = f.
\end{aligned}
$$

Exercise 1. *Use the simplex method to solve the two-person zero-sum game with game matrix*

$$
\begin{pmatrix}
-1 & 1 & 3 \\
1 & -3 & 2 \\
3 & 0 & -1
\end{pmatrix}.
$$

Solution. Step 1. Add 3 to each entry, we get

$$\begin{pmatrix} 2 & 4 & 6 \\ 4 & 0 & 5 \\ 6 & 3 & 2 \end{pmatrix}.$$

Step 2. Set up the tableau as

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & -1 \\ \hline x_1 & 2 & 4 & 6 & 1 \\ x_2 & 4 & 0 & 5 & 1 \\ x_3 & 6 & 3 & 2 & 1 \\ \hline -1 & 1 & 1 & 1 & 0 \end{array}.$$

Step 3. Apply pivoting operations, we have

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & -1 \\ \hline x_1 & 2 & 4 & 6 & 1 \\ x_2 & 4 & 0 & 5 & 1 \\ x_3 & 6^* & 3 & 2 & 1 \\ \hline -1 & 1 & 1 & 1 & 0 \end{array} \to \begin{array}{c|ccc|c} & x_3 & y_2 & y_3 & -1 \\ \hline x_1 & -\frac{1}{3} & 3^* & \frac{16}{3} & \frac{2}{3} \\ x_2 & -\frac{2}{3} & -2 & \frac{11}{3} & \frac{1}{3} \\ y_1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \hline -1 & -\frac{1}{6} & \frac{1}{2} & \frac{2}{3} & -\frac{1}{6} \end{array} \to$$

$$\to \begin{array}{c|ccc|c} & x_3 & x_1 & y_3 & -1 \\ \hline y_2 & -\frac{1}{9} & \frac{1}{3} & \frac{16}{9} & \frac{2}{9} \\ x_2 & -\frac{8}{9} & \frac{2}{3} & \frac{115}{27} & \frac{11}{27} \\ y_1 & \frac{2}{9} & -\frac{1}{6} & -\frac{5}{9} & \frac{1}{18} \\ \hline -1 & -\frac{1}{9} & -\frac{1}{6} & -\frac{2}{9} & -\frac{5}{18} \end{array}.$$

Let $d = \frac{5}{18}$. Then the value of the game is $v = \frac{1}{d} - 3 = \frac{3}{5}$. Since the basic solution is

$$\begin{aligned} x_3 &= \frac{1}{9} & y_6 &= 0 \\ x_1 &= \frac{1}{6} & y_4 &= 0 \\ y_3 &= 0 & x_6 &= \frac{2}{9} \\ y_2 &= \frac{2}{9} & x_5 &= 0 \\ x_2 &= 0 & y_5 &= \frac{11}{27} \\ y_1 &= \frac{1}{18} & x_4 &= 0 \end{aligned}$$

We have the maximin strategy for the row player is

$$\boldsymbol{p} = \frac{1}{d}(x_1, x_2, x_3) = \frac{18}{5}(\frac{1}{6}, 0, \frac{1}{9}) = (\frac{3}{5}, 0, \frac{2}{5}),$$

and the minimax strategy for the column player is

$$\boldsymbol{q} = \frac{1}{d}(y_1, y_2, y_3) = \frac{18}{5}(\frac{1}{18}, \frac{2}{9}, 0) = (\frac{1}{5}, \frac{4}{5}, 0).$$

Exercise 2. *Let A be an $m \times n$ matrix. Let*

$$C = \mathrm{conv}(\{\mathrm{a}_1, \cdots, \mathrm{a}_\mathrm{n}, \mathrm{e}_1, \cdots, \mathrm{e}_\mathrm{m}\})$$

be the convex hull of set $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_m\}$, where $\boldsymbol{a}_1^T, \cdots, \boldsymbol{a}_n^T$ are the column vectors of A and $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_m$ are the vectors in the standard basis of $\mathbb{R}^m$. Prove if C contains a point $(c, \cdots, c) \in \mathbb{R}^m$ with $c \leq 0$, then the value of A, $v(A) \leq c$.